1. You are going to play 2 games of chess with an opponent whom you have never played against before (for the sake of this problem). Your opponent is equally likely to be a beginner, intermediate, or a master. Depending on   
   (a) What is your probability of winning the first game?   
   (b) Congratulations: you won the first game! Given this information, what is the probability that you will also win the second game   
   (c) Explain the distinction between assuming that the outcomes of the games are independent and assuming that they are conditionally independent given the opponent’s skill level. Which of these assumptions seems more reasonable, and why?

Answer:

**Probability of Winning the First Game**

**Definitions and Given Information:**

* **Opponent's skill levels:**
  + Beginner
  + Intermediate
  + Master
* **Probabilities of each skill level:**

P(Beginner)=13P(\text{Beginner}) = \frac{1}{3}P(Beginner)=31​ P(Intermediate)=13P(\text{Intermediate}) = \frac{1}{3}P(Intermediate)=31​ P(Master)=13P(\text{Master}) = \frac{1}{3}P(Master)=31​

* **Winning probabilities based on skill level:**
  + Beginner: P(Win | Beginner)=0.9P(\text{Win | Beginner}) = 0.9P(Win | Beginner)=0.9
  + Intermediate: P(Win | Intermediate)=0.5P(\text{Win | Intermediate}) = 0.5P(Win | Intermediate)=0.5
  + Master: P(Win | Master)=0.1P(\text{Win | Master}) = 0.1P(Win | Master)=0.1

**Calculate the overall probability of winning the first game using the Law of Total Probability:**

P(Win)=P(Win | Beginner)×P(Beginner)+P(Win | Intermediate)×P(Intermediate)+P(Win | Master)×P(Master)P(\text{Win}) = P(\text{Win | Beginner}) \times P(\text{Beginner}) + P(\text{Win | Intermediate}) \times P(\text{Intermediate}) + P(\text{Win | Master}) \times P(\text{Master})P(Win)=P(Win | Beginner)×P(Beginner)+P(Win | Intermediate)×P(Intermediate)+P(Win | Master)×P(Master) P(Win)=(0.9×13)+(0.5×13)+(0.1×13)P(\text{Win}) = (0.9 \times \frac{1}{3}) + (0.5 \times \frac{1}{3}) + (0.1 \times \frac{1}{3})P(Win)=(0.9×31​)+(0.5×31​)+(0.1×31​) P(Win)=0.3+0.1667+0.0333=0.50P(\text{Win}) = 0.3 + 0.1667 + 0.0333 = 0.50P(Win)=0.3+0.1667+0.0333=0.50

**Final Answer for (a):**

The probability of winning the first game is **0.50** or **50%**.

**2. Probability of Winning the Second Game Given Winning the First Game**

Let’s use Bayes' Theorem to determine the probability that you will win the second game given that you won the first game.

**Define events:**

* W1W\_1W1​: Winning the first game.
* W2W\_2W2​: Winning the second game.

**We need to find P(W2∣W1)P(W\_2 | W\_1)P(W2​∣W1​), the probability of winning the second game given that you won the first game.**

**Using Bayes' Theorem:**

P(W2∣W1)=P(W1∣W2)×P(W2)P(W1)P(W\_2 | W\_1) = \frac{P(W\_1 | W\_2) \times P(W\_2)}{P(W\_1)}P(W2​∣W1​)=P(W1​)P(W1​∣W2​)×P(W2​)​

**Given the opponent’s skill level is conditional on winning:**

* Let SSS be the opponent's skill level.

To simplify:

P(W2∣W1)=P(W2∣S) where S is the skill level given W1P(W\_2 | W\_1) = P(W\_2 | S) \text{ where } S \text{ is the skill level given } W\_1P(W2​∣W1​)=P(W2​∣S) where S is the skill level given W1​

Calculate the posterior probabilities:

P(S∣W1)=P(W1∣S)×P(S)P(W1)P(S | W\_1) = \frac{P(W\_1 | S) \times P(S)}{P(W\_1)}P(S∣W1​)=P(W1​)P(W1​∣S)×P(S)​

**Compute probabilities for each skill level given winning the first game:**

* **Beginner:**

P(S=Beginner∣W1)=P(W1∣Beginner)×P(Beginner)P(W1)=0.9×130.50=0.30.50=0.60P(S = \text{Beginner} | W\_1) = \frac{P(W\_1 | \text{Beginner}) \times P(\text{Beginner})}{P(W\_1)} = \frac{0.9 \times \frac{1}{3}}{0.50} = \frac{0.3}{0.50} = 0.60P(S=Beginner∣W1​)=P(W1​)P(W1​∣Beginner)×P(Beginner)​=0.500.9×31​​=0.500.3​=0.60

* **Intermediate:**

P(S=Intermediate∣W1)=P(W1∣Intermediate)×P(Intermediate)P(W1)=0.5×130.50=0.16670.50=0.333P(S = \text{Intermediate} | W\_1) = \frac{P(W\_1 | \text{Intermediate}) \times P(\text{Intermediate})}{P(W\_1)} = \frac{0.5 \times \frac{1}{3}}{0.50} = \frac{0.1667}{0.50} = 0.333P(S=Intermediate∣W1​)=P(W1​)P(W1​∣Intermediate)×P(Intermediate)​=0.500.5×31​​=0.500.1667​=0.333

* **Master:**

P(S=Master∣W1)=P(W1∣Master)×P(Master)P(W1)=0.1×130.50=0.03330.50=0.0667P(S = \text{Master} | W\_1) = \frac{P(W\_1 | \text{Master}) \times P(\text{Master})}{P(W\_1)} = \frac{0.1 \times \frac{1}{3}}{0.50} = \frac{0.0333}{0.50} = 0.0667P(S=Master∣W1​)=P(W1​)P(W1​∣Master)×P(Master)​=0.500.1×31​​=0.500.0333​=0.0667

**Probability of winning the second game given each skill level:**

P(W2∣Beginner)=0.9P(W\_2 | \text{Beginner}) = 0.9P(W2​∣Beginner)=0.9 P(W2∣Intermediate)=0.5P(W\_2 | \text{Intermediate}) = 0.5P(W2​∣Intermediate)=0.5 P(W2∣Master)=0.1P(W\_2 | \text{Master}) = 0.1P(W2​∣Master)=0.1

**Overall probability of winning the second game:**

P(W2∣W1)=P(W2∣S=Beginner)×P(S=Beginner∣W1)+P(W2∣S=Intermediate)×P(S=Intermediate∣W1)+P(W2∣S=Master)×P(S=Master∣W1)P(W\_2 | W\_1) = P(W\_2 | S = \text{Beginner}) \times P(S = \text{Beginner} | W\_1) + P(W\_2 | S = \text{Intermediate}) \times P(S = \text{Intermediate} | W\_1) + P(W\_2 | S = \text{Master}) \times P(S = \text{Master} | W\_1)P(W2​∣W1​)=P(W2​∣S=Beginner)×P(S=Beginner∣W1​)+P(W2​∣S=Intermediate)×P(S=Intermediate∣W1​)+P(W2​∣S=Master)×P(S=Master∣W1​) P(W2∣W1)=(0.9×0.60)+(0.5×0.333)+(0.1×0.0667)P(W\_2 | W\_1) = (0.9 \times 0.60) + (0.5 \times 0.333) + (0.1 \times 0.0667)P(W2​∣W1​)=(0.9×0.60)+(0.5×0.333)+(0.1×0.0667) P(W2∣W1)=0.54+0.1665+0.0067=0.7132P(W\_2 | W\_1) = 0.54 + 0.1665 + 0.0067 = 0.7132P(W2​∣W1​)=0.54+0.1665+0.0067=0.7132

**Final Answer for (b):**

The probability of winning the second game given that you won the first game is approximately **0.7132** or **71.32%**.

**3. Independence vs. Conditional Independence**

**Independence:**

* Assuming the outcomes of the games are independent means the result of the second game does not depend on the outcome of the first game.

**Conditional Independence:**

* Assuming the outcomes are conditionally independent given the opponent’s skill level means the outcome of the second game depends on the skill level, but once you know the skill level, the result of the second game does not provide additional information about the result of the first game.

**Which Assumption is More Reasonable?**

* **Conditional Independence** is more reasonable because the skill level of the opponent affects the probability of winning both games. Once you know the opponent's skill level, the outcomes of the games are influenced by that skill level. Assuming independence would ignore this dependence and may not reflect the true nature of the games.